DETERMINING THE STATISTICAL CHARACTERISTICS OF A LINEAR DYNAMIC SYSTEM FROM MEASUREMENTS OF ITS MOTION

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In constructing mathematical descriptions of mechanical systems, automatic control systems, etc. from measurements of the processes at the input and output of the system (i.e. in solving the identification problem) one often meets with the fact that the process at the system output is random even with a strictly determined input signal (provided the process at the system output is identically equal to zero when there is no signal at the input). This fact testifies to the "stochasticity" of the system under consideration; a sufficiently complete mathematical model of such a system must contain parameters which are random functions of time, and the identification of such a system must include the determination of the statistical characteristics of these variations of the parameters. One of the uses of these characteristics lies in estimating the quality and reliability of systems (e.g. in those cases where the "nominal' values of the parameters lie near the boundary of the stability domain).

Biomechanics affords several examples of stochastic systems. Among them are the problem of the effect of vibrations on the human organism [1] and the problem of constructing a mathematical model of a human operator as a link in an automatic control system [2].

In the present paper we consider a system described by an ordinary differential equation whose coefficients are time-independent and time-independently constrained random functions; the order of the equation is assumed to be known. A sinusoidal force is applied to the system. In accordance with the approximate solution of the "straightforward" problem of statistical dynamics [3] we construct an iterative process for finding the unknown moment functions of the system coefficients from the measured moment functions of the process at the output. As an example we consider a second-order system with a randomly varying damping coefficient.

1. Let us investigate the steady motion of a system described by the differential equation

$$[Q_0(p) + \mu Q_1(t, p)]\, x = [P_0(p) + \mu P_1(t, p)]\, y, \qquad y = \cos \omega_0 t$$

$$Q_0(p) = p^n + \sum_{k=0}^{n-1} a_k p^k, \qquad P_0(p) = \sum_{k=0}^{m} c_k p^k, \qquad p = \frac{d}{dt} \qquad (m < n)$$

$$Q_1(t, p) = \sum_{k=0}^{n-1} a_k \xi_k(t)\, p^k, \qquad P_1(t, p) = \sum_{k=0}^{m} c_k \eta_k(t)\, p^k \tag{1.1}$$

Here μ, a_k, and c_k are unknown constants; $\xi_k(t)$ and $\eta_k(t)$ are time-independent and time-independent and time-independently constrained centered ergodic random functions whose statistical characteristics must be determined; the roots of the polynomial Q_0 are assumed to be distinct and to have negative real parts. We assume that Eq. (1.1) has a particular solution $x(t)$ all of whose moments are finite (the problem is meaningless otherwise), and that realization of the process $x(t)$ for various values of ω_0 are known.

Let us assume that the parameter μ is small and write out the function $x(t)$ in series form

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \ldots \tag{1.2}$$

We then obtain the following expansions for the average value and correlation function of the process $x(t)$ (the square brackets will generally be used to denote averaging over the set of realizations):

$$\langle x(t)\rangle = x_0(t) + \mu^2\langle x_2(t)\rangle + \ldots \tag{1.3}$$

$$\langle[x(t) - \langle x(t)\rangle][x(t+\tau) - \langle x(t+\tau)\rangle]\rangle = \mu^2\langle x_1(t)\, x_1(t+\tau)\rangle + \\ +\mu^3[\langle x_1(t)\, x_2(t+\tau)\rangle + \langle x_2(t) x_1(t+\tau)\rangle] + \ldots \tag{1.4}$$

(as will become evident below, $\langle x_1(t)\rangle = 0$ by virtue of the fact that the average values of the random functions $\xi(t)$ and $\eta(t)$ are equal to zero). Similar expansions are valid for the central moment functions of higher orders; the first term of the series for the central k-th order moment of the process $x(t)$ is simply the corresponding k-th order moment of the process $x_1(t)$ taken with its coefficient μ^k. The convergence of these expansions for sufficiently small μ can be proved on the basis of the condition of time-independence and boundedness of the functions $\xi(t)$ and $\eta(t)$. In solving the "straightforward" problem where the moments of these functions are known we can also find a lower estimate of the convergence interval. Convergence cannot be estimated in advance in the problem under consideration (more precisely, only rough qualitative estimates are possible), and this estimate is established in the course of direct computation of the successive approximations.

Substituting series (1.2) into Eq. (1.1), we obtain the infinite system of equations

$$Q_0(p)\, x_0 = P_0(p)\, y, \qquad Q_0(p)\, x_1 = -\, Q_1(t, p)\, x_0 + P_1(t, p)\, y \\ Q_0(p)\, x_i = -\, Q_1(t, p)\, x_{i-1} \qquad (i \geqslant 2) \tag{1.5}$$

The particular solutions of these equations corresponding to the steadystate oscillations of the system can be written as

$$x_0(t) = b_0 \sin(\omega_0 t + \varphi_0) \tag{1.6}$$

$$x_1(t) = \int_{-\infty}^{t} G(t-t')\,[-Q_1(t', p')\, x_0(t') + P_1(t', p')\, y(t')]\, dt' \quad \left(p' = \frac{d}{dt'}\right) \tag{1.7}$$

$$x_i(t) = -\int_{-\infty}^{t} G(t-t')\, Q_1(t', p')\, x_{i-1}(t')\, dt' \qquad (i \geqslant 2) \tag{1.8}$$

where $G(\theta)$ is Green's function for the operator Q_0

$$G(\theta) = \sum_{s=1}^{n} \frac{\exp q_s\theta}{Q_0'(q_s)}, \qquad Q_0'(q) = \frac{dQ_0}{dq} \tag{1.9}$$

(q_s denote the roots of the polynomial Q_0). On the basis of relations (1.6) to (1.8) we can apply the averaging operation and express the average value and central moments of the process $x(t)$ as an infinite series whose terms are determined by the moments of the random functions $\xi(t)$ and $\eta(t)$. Specifically, the first term of series (1.4) is given by

$$\mu^2\langle x_1(t)\, x_1(t+\tau)\rangle = \mu^2 \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} G(t-t')\, G(t+\tau-t'')\,\langle[-Q_1(t', p')\, x_0(t') + \\ + P_1(t', p')\, y(t')]\,[-Q_1(t'', p'')\, x_0(t'') + P_1(t'', p'')\, y(t'')]\rangle\, dt'\, dt'' \tag{1.10}$$

Before turning to the "inverse" problem let us show that if series (1.2) converges in the mean and mean-square, then the process $x(t)$ can be expressed in the form $x(t) = f(t)\cos\omega_0 (t) + g(t)\sin\omega_0 t$, where $f(t)$ and $g(t)$ are time-independent and time-independently constrained (in the broad sense) random processes. In fact, on the basis of (1.6) to (1.8) we can write the k-th term of series (1.2) as a linear combination of terms of the type

$$z_k(t)\,e^{-i\omega_0 t} + z_{k*}(t)\,e^{i\omega_0 t}, \qquad z_k(t) = \int\limits_{\Gamma_k} G(t-\theta_1)\,e^{i\omega_0(t-\theta_1)} G(\theta_1-\theta_2) \times$$

$$\times\, e^{i\omega_0(\theta_1-\theta_2)} \ldots G(\theta_{k-1}-\theta_k)\,e^{i\omega_0(\theta_{k-1}-\theta_k)}\,\xi_i(\theta_1)\,\xi_j(\theta_2)\ldots\xi_l(\theta_k)\,d\Gamma_k \tag{1.11}$$

and similar integrals containing the functions $\eta(t)$ and products of the functions $\xi(t)$ and $\eta(t)$. Here Γ_k is the domain of k-dimensional space defined by the condition $-\infty < \theta_k < \theta_{k-1} < \ldots < \theta_1 < t$; the asterisk denotes the complex conjugate. From (1.11) we can readily see that the time independence and time-independent constrain (in the narrow sense) of the processes $\xi(t)$ and $\eta(t)$ imply the time independence and time-independent constraint of the processes $z_j(t)$, $z_k(t)$, $z_{j*}(t)$, and $z_{k*}(t)$ (in the broad sense), so that the mean-square convergence of series (1.2) is sufficient to validate the above statement. This implies, specifically, that the values of the process $x(t)$ at the instants t_k satisfying the conditions

$$t_k - t_l = (2\pi/\omega_0)(k-l), \quad k-l = 0,\ \pm 1,\ \pm 2, \ldots \tag{1.12}$$

coincide with the values of some random process which is time-independent in the broad sense. Similarly we can show that the higher-order moments of this process likewise depend only on the differences between the values of t (provided series (1.2) converges for the corresponding moments). However, the ergodicity of this process is assumed by hypothesis; if this condition is not fulfilled, then the problem of identification from the family of realizations of $x(t)$ dependent on just a single parameter ω_0 becomes meaningless.

2. Let us assume that we know the average value and central moments of the process $x(t)$ computed for the values $t = t_k, t_l, \ldots$ satisfying condition (1.12). Relations (1.6) (the solution of the first Eq. of (1.5)) and (1.10), and the similar relations for the higher-order moments of the process $x_1(t)$ will be our basis for determining the statistical characteristics of the random functions $\xi(t)$ and $\eta(t)$. Since the parameters b_0 and φ_0 of the function $x_0(t)$ and the moments of the process $x_1(t)$ are unknown, we shall find them by successive approximations from the known sums of infinite series (1.3) and (1.4) and similar series for the higher-order central moments of the process $x(t)$. We begin by setting

$$x_0(t) = \langle x(t)\rangle = [\langle f(t)\rangle^2 + \langle g(t)\rangle^2]^{1/2} \sin[\omega_0 t + \operatorname{arc\,tg}(\langle g(t)\rangle / \langle f(t)\rangle)] \tag{2.1}$$

The quantities $\langle f(t)\rangle$ and $\langle g(t)\rangle$ are here determined on the basis of $x(t)$ measured for two families of points t_k satisfying conditions (1.12) and corresponding to the zeros of the functions $\sin\omega_0 t$ and $\cos\omega_0 t$. As a result we arrive at the problem of determining the coefficients of the first Eq. of (1.5) from its solution (1.6) where b_0 and φ_0 are known functions of the parameter ω_0. As is shown in [4] this problem reduces to the approximation of the function $b_0(\zeta)\exp[i\varphi_0(\zeta)]$ of the complex variable $\zeta = \varkappa + i\omega_0$ whose values are specified on the imaginary axis by a function which is a ratio of polynomials of degrees m and n and has poles in the left half-plane only; the zeros and poles of this function determine the roots of the polynomials $P_0(q)$ and $Q_0(q)$, respectively. Some methods for practical construction of the solution will be found in [5], for example.

Now let us equate the correlation function of the process $x(t)$ to the first term of series (1.4) for S values ω_0, where S must not be smaller than the number of nonzero elements of the matrix of correlation functions $K(v)$ of the processes $\xi(t)$ and $\eta(t)$. Let us convert to the variables $u = t'' + t'$ and $v = t'' - t'$ and integrate over u in relation (1.10), in which a_k, c_k, b_0, φ_0, and $G(\theta)$ are now known. Substituting the known values of $\sin(\omega_0 t_k + \varphi_0)$ and $\cos(\omega_0 t_k + \varphi_0)$ into the result, we obtain a system of S integral first-order Fredholm integral equations in the function $K(v)$. In the general case such a system must be solved approximately, as is often done in identification problems [6]. We accomplish them by specifying functional relations for $K(v)$ which contain a finite number of unknown parameters γ determined by the method of least squares. However, it is sometimes possible (see the example in Section 3) to obtain directly the inversion formula which expresses $K(v)$ in terms of the known functions $R_1(\tau) = \mu^2\langle x_1(t_k)\,x_1(t_k+\tau)\rangle$.

The next step in the iteration process consists in refining the values of a_k and c_k by using two terms of the series in the exoression for $\langle x(t)\rangle$. To this end we substitute the known functions $K(v)$ into the relation for $\langle x_2(t)\rangle$ constructed on the basis of (1.7) and (1.8) (computations of $\langle x_2(t)\rangle$ for the special case $m = 0$ will be found in [3]). Setting $x_0(t) = \langle x(t)\rangle - \mu^2\langle x_2(t)\rangle$, we obtain refined values for b_0 and φ_0; returning to the solution of the first problem, we obtain improved values for a_k and c_k, and hence an improved function $G(\theta)$.

The general scheme of the iteration process we are proposing consists in the following. Each step involving the addition of terms of order μ^r begins with consideration of the relation for the r-th moment of the process $x_1(t)$ (as noted above, the aforementioned moment taken with the coefficient μ^r coincides with the corresponding central moment of the process $x(t)$). From this relation we obtain the values of the parameters which determine the r-th order moments of the processes $\xi(t)$ and $\eta(t)$ by a method similar to that just described for second-order moments. Then, having determined the terms of order μ^r appearing in the series for the s-th order central moments of the process $x(t)$ $(s < r)$, we can refine $x_0(t)$ and the s-th order moments of the process $x_1(t)$; we then refine the values of a_k and c_k and the values of the parameters which determine the s-th order moments of (the processes $\xi(t)$ and $\eta(t)$ $(2 \leqslant s \leqslant r-1)$. We note, however, that such a scheme using information on the higher-order moments of the process $x(t)$ is not the only acceptable one: refinement involving the addition of terms of order μ^r to the series for the s-th order moment of the process $x(t)$ $(s < r)$ can be effected directly by specifying (to within a finite number of parameters δ) the expressions relating the r-th order moments of the processes $\xi(t)$ and $\eta(t)$ to the lower-order moments. In this case the parameters δ are determined simultaneously with the refinement of the previously determined values of the parameters γ which determine the lower-order moments.

In the case of "quasiharmonic" systems which are reducible to standard form [7], the relation for $R_1(\tau)$ can be simplified substantially by first averaging over the period of the terms not containing the functions $\xi(t)$ and $\eta(t)$. The approximate relations obtained through this simplification are useful, for example, in the solution of problems on an analog computer by direct selection of the model parameters. A sample solution of the problem by this method appears below.

3. Let us consider, for example, the following second-order system with a randomly varying damping coefficients:

$$\frac{d^2x}{dt^2} + 2\alpha\,[1 + \mu\xi\,(t)]\,\frac{dx}{dt} + \Omega^2 x = a\cos\omega_0 t \tag{3.1}$$

The function $G(\theta)$ in this case is given by

$$G(0) = \omega_1^{-1}\exp(-\alpha\theta)\sin\omega_1\theta, \qquad \omega_1 = \sqrt{\Omega^2 - \alpha^2} \tag{3.2}$$

For simplicity we assume that the values of α and Ω are known and pose the problem of determining the correlation function $\mu^2 K(v)$ of the process $\mu\xi(t)$ from one realization of the process $x(t)$. The parameters b_0 and φ_0 in our case are given by

$$b_0 = a\;[(\Omega^2 - \omega_0^2)^2 + 4\alpha^2\omega_0^2]^{-1/2},\quad \varphi_0 = \operatorname{arc\;tg}\;[(\Omega^2 - \omega_0^2)\,/\,2\alpha\omega_0]$$

Constructing Expression (1.10), making use of (3.2), and integrating over the variable $u = t'' + t'$ (we are considering the case $\sin(\omega_0 t_k + \varphi_0) = 1$), we obtain

$$R_1(\tau) = {}^1\!/_4\,\mu^2(\alpha b_0\omega_0\,/\,\omega_1)^2\int\limits_{-\infty}^{\infty} K(v)\,[C^+\cos(\omega_0 + \omega_1)\,|v-\tau| + D^+\sin(\omega_0+\omega_1)\,|v-\tau| +$$

$$+ C^-\cos(\omega_0 - \omega_1)\,|v-\tau| - D^-\sin(\omega_0 - \omega_1)\,|v-\tau|]\exp(-\alpha\,|v-\tau|)\,dv \tag{3.3}$$

Here

$$C^{\pm} = 1\,/\,\alpha + \alpha\,\{[\alpha^2 + (\omega_0 \pm \omega_1)^2]^{-1} - (\alpha^2 + \omega_0^2)^{-1} - (\alpha^2 + \omega_1^2)^{-1}\}$$

$$D^{\pm} = \omega_1\,(\alpha^2 + \omega_1^2)^{-1} \pm \omega_0\,(\alpha^2 + \omega_0^2)^{-1} \mp (\omega_0 \pm \omega_1)\;[\alpha^2 + (\omega_0 \pm \omega_1)^2]^{-1}$$

In deriving relation (3.3) we made use of the evenness property of the function $K(v)$ and the condition $\exp(\pm i\omega_0\tau) = 1$.

We note that in this case first-order integral Eq. (3.3) can be solved directly for the unknown function $K(v)$ [8]. This solution is of the form

$$\mu^2 K(v) = \frac{4}{\sqrt{2\pi}}(\alpha b_0\omega_0/\omega_1)^{-2}\int_{-\infty}^{\infty}\Phi(\omega)\{[C^+\alpha + D^+(\omega_0+\omega_1-\omega)][\alpha^2+(\omega_0+\omega_1-\omega)^2]^{-1}+$$
$$+[C^+\alpha + D^+(\omega_0+\omega_1+\omega)][\alpha^2+(\omega_0+\omega_1+\omega)^2]^{-1}+$$
$$+[C^-\alpha - D^-(\omega_0-\omega_1+\omega)][\alpha^2+(\omega_0-\omega_1+\omega)^2]^{-1}+$$
$$+[C^-\alpha - D^-(\omega_0-\omega_1-\omega)][\alpha^2+(\omega_0-\omega_1-\omega)^2]^{-1}\}^{-1}e^{-i\omega v}\,d\omega \tag{3.4}$$

where $\Phi(\omega)$ is the spectral density of the "sampling" process $\mu x_1(t_k)$,

$$\Phi(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}R_1(\tau)e^{i\omega\tau}\,d\tau$$

Here we assume that the function $K(v)$ satisfies the requirements formulated in [8] which guarantee the uniqueness and correctness of the resulting solution. We need merely bear in mind that the values of $R_1(\tau)$ are known only for the discrete set of points $\tau_i = 2\pi i/\omega_0$; the exactness of the resulting estimates will not be considered.

We note that the resulting solution can also be used in the case where some unmeasurable random disturbance of the form $\mu\zeta(t)$ is added to the useful signal defined by the right side of Eq. (3.1). In fact, we can readily show that the components of the correlation function of the process $x_1(t)$ which correspond to this disturbance do not contain the factor b_0^2. Hence, these components can be isolated by varying the amplitude a of the useful signal, whereupon the function $K(v)$ can be determined by the method described above.

Now let us illustrate a result obtained through the averaging method (see [9], where a similar technique is used to investigate several other "straightforward" problems). For simplicity let us consider the case of exact resonance ($\omega_0 = \Omega$). Setting $\alpha = \mu\alpha_1$ and $a = \mu a_1$ and carrying out the substitution of variables

$$x(t) = b(t)\sin[\omega_0 t + \varphi(t)],\quad dx(t)/dt = \omega_0 b(t)\cos[\omega_0 t + \varphi(t)] \tag{3.5}$$

we convert from (3.1) to equations in standard form in the functions $b(t)$ and $\varphi(t)$. After the terms not containing $\xi(t)$ in these expressions have been averaged over the period $2\pi/\omega_0$, the solutions for the average amplitude and phase, denoted by b and φ as before, are sought in series form,

$$b(t) = b_0 + \mu b_1(t) + \ldots,\quad \varphi(t) = \varphi_0 + \mu\varphi_1(t) + \ldots$$

The final expression for the first approximation of the correlation function of the random component μb_1 of the amplitude is

$$\mu^2\langle b_1(t)\,b_1(t+\tau)\rangle = {}^1/_2\,\mu^2\alpha b_0^2\int_{-\infty}^{\infty}K(v)\,e^{-\alpha|v-\tau|}(1+{}^1/_2\cos 2\omega_0 v)\,dv \tag{3.6}$$

Comparing (3.6) with "exact first approximation" (3.3), we can estimate the domain of applicability of the averaging method. It is easy to verify that these expressions can be made to coincide by rejecting terms of higher order of smallness in α/Ω in the case $\omega_0 = \Omega$ and setting

$$\cos(\omega_0-\omega_1)\tau \approx 1,\qquad \sin(\omega_0-\omega_1)\tau \approx 0 \tag{3.7}$$
$$\cos(\omega_0-\omega_1)v \approx 1,\qquad \sin(\omega_0-\omega_1)v \approx 0 \tag{3.8}$$

The requirement that α/Ω be small is self-evident. Fulfillment of this condition guarantees fulfillment of condition (3.8). As regards conditions (3.7), they are equivalent to the condition $\Omega\tau \ll \pi(\Omega/\alpha)^2$ and reflect to the known [7] fact that the averaging method affords a good approximation only over time intervals of order $1/\mu$.

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